

Hydrodynamics of R-charged black holes

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ABSTRACT: We consider hydrodynamics of $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills plasma at a nonzero density of R -charge. In the regime of large N_c and large 't Hooft coupling the gravity dual description involves an asymptotically Anti-de Sitter five-dimensional charged black hole solution of Behrnd, Cvetič and Sabra. We compute the shear viscosity as a function of chemical potentials conjugated to the three $U(1) \subset SO(6)_R$ charges. The ratio of the shear viscosity to entropy density is independent of the chemical potentials and is equal to $1/4\pi$. For a single charge black hole we also compute the thermal conductivity, and investigate the critical behavior of the transport coefficients near the boundary of thermodynamic stability.

KEYWORDS: Black Holes in String Theory, AdS-CFT Correspondence, Thermal Field Theory.

Contents

1. Introduction	1
2. The R-charged black hole background	2
3. Shear viscosity	5
4. Shear viscosity from the diffusion pole	7
4.1 The single charge black hole background	7
4.2 The correlators	9
4.3 The diffusion pole	11
5. Thermal conductivity	11
6. Critical behavior of transport coefficients	12
7. Conclusion	12
A. Rescaling the black hole solution	14
B. Relativistic hydrodynamics at finite chemical potential	15

1. Introduction

Recent studies of strongly coupled thermal gauge theories in the framework of the gauge-gravity duality ([1–3], for a review see [4]) suggest that in all those theories in the regime described by gravity duals the ratio of the shear viscosity to volume entropy density is universal and equal to $1/4\pi$ [5–7]. As this result was obtained assuming zero densities of conserved charges, a natural question to ask is what happens when the chemical potentials conjugated to these charges are turned on.

The simplest ten-dimensional gravitational background corresponding to a non-zero chemical potential in a dual four-dimensional finite-temperature field theory is the one of spinning near-extremal three-branes [8–10]. The number of independent commuting angular momenta that can be given to the three-branes is equal to the rank $r = 3$ of the isometry group $SO(6)$ of the space transverse to the branes. Upon dimensional reduction on S^5 one obtains the background corresponding to the five-dimensional asymptotically AdS black hole of the Reissner-Nordström type with three $U(1)$ charges proportional to the angular momenta of the branes [10]. This background was found by Behrnd, Cvetič and Sabra [11] as a particular solution to the equations of motion of the five-dimensional

$\mathcal{N} = 2$ gauged supergravity. The solution corresponding to the dimensional reduction of the spinning three-brane metric has a translationally invariant horizon. (It can also be regarded as a black hole with a spherical horizon in an infinite volume limit.)

In the AdS/CFT correspondence the isometry group of S^5 is interpreted as the R -symmetry group of the dual $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. Three independent chemical potentials μ_i can be introduced as the Lagrange multipliers to the three $U(1)$ charges in the Cartan subalgebra of $SO(6)_R$. Thermodynamics of the R-charged black holes in the context of the AdS/CFT correspondence was originally studied in [8, 12], [9, 10]. One feature relevant for our discussion is the thermodynamic instability¹ occurring for black holes with excessively large charge (or equivalently for the three-branes rotating too fast) [8, 12]. The instability may signal the onset of a phase transition. However, both in gravity and in the dual strongly coupled field theory picture it is not clear what the new phase might be.

In this paper we study the hydrodynamic regime of the four-dimensional $\mathcal{N} = 4$ $SU(N_c)$ SYM theory with three non-zero chemical potentials in the limit of large N_c and large 't Hooft coupling $g_{YM}^2 N_c$. Using Kubo formula, we compute the shear viscosity as a function of the three charges (or chemical potentials) and show that for any values of the charges in the thermodynamic stability domain the ratio of the viscosity to entropy density is equal to $1/4\pi$. For a technically simpler case of a single charge black hole we explicitly compute thermal correlation functions of the stress-energy tensors and R-currents in the shear channel in the hydrodynamic approximation. The correlators exhibit a diffusion pole with the dispersion relation that confirms the result for the shear viscosity found from the Kubo formula. We also compute thermal conductivity. The ratio of thermal conductivity and shear viscosity obeys a simple relation reminiscent of the Wiedemann-Franz law for the ratio of the thermal conductivity to the electrical conductivity. Finally, we investigate the behavior of the transport coefficients near the boundary of thermodynamic stability and compute the corresponding critical exponent.

The paper is organized as follows. We review the STU-model solution of Behrnd, Cvetič and Sabra in section 2. The shear viscosity for the three-charge black hole solutions is computed in section 3. Starting from section 4, we specialize to the single-charge background: in section 4 we compute the correlation functions of the shear mode components of the stress-energy tensors and R-currents in the hydrodynamic approximation, in section 5 we obtain the thermal conductivity, and in section 6 we discuss the critical behavior of the transport coefficients. For convenience of the reader, in appendix A we outline the rescaling procedure necessary to obtain an asymptotically AdS charged black hole with a translationally invariant horizon from the black hole with a spherical horizon. In appendix B we review relativistic hydrodynamics at finite chemical potential.

2. The R-charged black hole background

The STU-model solution to equations of motion of $D = 5$ $\mathcal{N} = 2$ gauged supergravity was found by Behrnd, Cvetič and Sabra [11]. The relevant part of the gauged supergravity

¹The related instability of the gravitational background was studied in [13].

effective Lagrangian is

$$\frac{\mathcal{L}}{\sqrt{-g}} = R + \frac{2}{L^2}\mathcal{V} - \frac{1}{2}G_{ij}F_{\mu\nu}^i F^{\mu\nu j} - G_{ij}\partial_\mu X^i \partial^\mu X^j + \frac{1}{24\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma\lambda}\epsilon_{ijk}F_{\mu\nu}^i F^{\rho\sigma j} A_\lambda^k, \quad (2.1)$$

where $F_{\mu\nu}^i$, $i = 1, 2, 3$ are the field-strength tensors of the three Abelian gauge fields, X^i are three real scalar fields subject to the constraint $X^1 X^2 X^3 = 1$. The metric on the scalar manifold is given by

$$G_{ij} = \frac{1}{2}\text{diag} [(X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}].$$

The scalar potential is

$$\mathcal{V} = 2 \sum_{i=1}^3 \frac{1}{X^i}.$$

The three-charge non-extremal STU solution is specified by the following background values of the metric

$$ds^2 = -\mathcal{H}^{-2/3} f_k dt^2 + \mathcal{H}^{1/3} (f_k^{-1} dr^2 + r^2 d\Omega_{3,k}^2), \quad (2.2)$$

$$f_k = k - \frac{m_k}{r^2} + \frac{r^2}{L^2}\mathcal{H}, \quad H_i = 1 + \frac{q_i}{r^2}, \quad \mathcal{H} = H_1 H_2 H_3, \quad (2.3)$$

as well as the scalar and the gauge fields

$$X^i = \frac{\mathcal{H}^{1/3}}{H_i}, \quad A_t^i = \sqrt{\frac{kq_i + m_k}{q_i}} (1 - H_i^{-1}). \quad (2.4)$$

Here m_k is the non-extremality parameter, and q_i , $i = 1, 2, 3$ are related to physical charges characterizing the background. The parameter k determines the spatial curvature of $d\Omega_{3,k}^2$: $k = 1$ corresponds to the metric on the three-sphere of unit radius, $k = 0$ - to the metric on \mathbb{R}^3 . It was shown in [10] that the $k = 0$ solution arises as the Kaluza-Klein reduction on S^5 of the ten-dimensional metric describing spinning near-extremal three-branes. The three R-charges q_i are related to the three independent angular momenta in ten dimensions. Since, strictly speaking, hydrodynamic regime is meaningful only in the case of a translationally-invariant horizon, in this paper we set² $k = 0$ and

$$d\Omega_{3,0}^2 \rightarrow \frac{1}{L^2} (dx^2 + dy^2 + dz^2).$$

Introducing the new radial coordinate $u = r_+^2/r^2$, where r_+ is the largest root of the equation $f(r) = 0$, we write the background fields in the form

$$ds_5^2 = -\mathcal{H}^{-2/3} \frac{(\pi T_0 L)^2}{u} f dt^2 + \mathcal{H}^{1/3} \frac{(\pi T_0 L)^2}{u} (dx^2 + dy^2 + dz^2) + \mathcal{H}^{1/3} \frac{L^2}{4fu^2} du^2, \quad (2.5)$$

$$f(u) = \mathcal{H}(u) - u^2 \prod_{i=1}^3 (1 + \kappa_i), \quad H_i = 1 + \kappa_i u, \quad \kappa_i \equiv \frac{q_i}{r_+^2}, \quad T_0 = r_+/\pi L^2. \quad (2.6)$$

²The background with a translationally-invariant horizon and related thermodynamics can also be obtained by taking an infinite volume limit of the $k = 1$ solution (see Appendix A).

The scalar fields and the gauge fields are given by³

$$X^i = \frac{\mathcal{H}^{1/3}}{H_i(u)}, \quad A_t^i = \frac{\tilde{\kappa}_i \sqrt{2}u}{LH_i(u)}, \quad \tilde{\kappa}_i = \frac{\sqrt{q_i}}{L} \prod_{i=1}^3 (1 + \kappa_i)^{1/2}. \quad (2.7)$$

The Hawking temperature of the background (2.5) is given by

$$T_H = \frac{2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1 \kappa_2 \kappa_3}{2\sqrt{(1 + \kappa_1)(1 + \kappa_2)(1 + \kappa_3)}} T_0. \quad (2.8)$$

The volume density of the Bekenstein-Hawking entropy is

$$s = \frac{A_H}{4G_5 V_3} = \frac{\pi^2 N^2 T_0^3}{2} \prod_{i=1}^3 (1 + \kappa_i)^{1/2}, \quad (2.9)$$

where V_3 is the spatial volume along the three infinite dimensions of the horizon, $G_5 = \pi L^3 / 2N^2$. The energy density and pressure are given by

$$\varepsilon = \frac{3\pi^2 N^2 T_0^4}{8} \prod_{i=1}^3 (1 + \kappa_i), \quad (2.10)$$

$$P = \frac{\pi^2 N^2 T_0^4}{8} \prod_{i=1}^3 (1 + \kappa_i). \quad (2.11)$$

The densities of physical charges are

$$\rho_i = \frac{\pi}{4G_5} \frac{r_+^2 \tilde{\kappa}_i L}{\sqrt{2}V_3} = \frac{\pi N^2 T_0^3}{8} \sqrt{2\kappa_i} \prod_{l=1}^3 (1 + \kappa_l)^{1/2}. \quad (2.12)$$

The chemical potentials conjugated to ρ_i are defined as

$$\mu_i = A_t^i(u) \Big|_{u=1} = \frac{\pi T_0 \sqrt{2\kappa_i}}{(1 + \kappa_i)} \prod_{l=1}^3 (1 + \kappa_l)^{1/2}. \quad (2.13)$$

For the grand canonical ensemble, where the system in thermodynamic equilibrium is characterized by the values of temperature and chemical potentials regulating its interaction with the surrounding heat bath, the appropriate thermodynamic potential is the Gibbs potential Ω ,

$$\Omega/V_3 = -P = \varepsilon - T_H s - \sum_{i=1}^3 \mu_i \rho_i. \quad (2.14)$$

The first law of thermodynamics reads

$$dP = s dT_H + \sum_{i=1}^3 \rho_i d\mu_i. \quad (2.15)$$

³We change the normalization of the gauge fields by a factor of $\sqrt{2}/L$. This normalization is used in the rest of the paper.

A stable thermodynamic equilibrium is determined by the conditions

$$(\delta\Omega)_{T,\mu_i \text{ fixed}} = 0, \quad (\delta^2\Omega)_{T,\mu_i \text{ fixed}} > 0. \quad (2.16)$$

The stability condition (2.16) translates into the equation

$$\det \left(\frac{\partial^2 \varepsilon(s, \rho_i)}{\partial s \partial \rho_i} \right) > 0. \quad (2.17)$$

Since $\kappa_i = 8\pi^2 \rho_i^2 / s^2$ and

$$\varepsilon(s, \rho_i) = \frac{3s^{4/3}}{2(2\pi N)^{2/3}} \prod_{i=1}^3 \left(1 + \frac{8\pi^2 \rho_i^2}{s^2} \right)^{1/3},$$

the condition of thermodynamic stability implies the following constraint on κ_i

$$2 - \kappa_1 - \kappa_2 - \kappa_3 - \kappa_1 \kappa_2 \kappa_3 > 0. \quad (2.18)$$

3. Shear viscosity

The simplest way to compute shear viscosity from the dual gravity background is to use Kubo formula. Kubo formula relates the shear viscosity to the correlation function of the stress-energy tensor at zero spatial momentum,

$$\eta = \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int dt d\mathbf{x} e^{i\omega t} \langle [T_{xy}(x), T_{xy}(0)] \rangle = - \lim_{\omega \rightarrow 0} \frac{\text{Im} G_{xyxy}(\omega, 0)}{\omega}, \quad (3.1)$$

where the retarded Green's function for the components of the stress-energy tensor is defined as

$$G_{\mu\nu\lambda\rho}(\omega, \mathbf{q}) = -i \int d^4x e^{-iq \cdot x} \theta(t) \langle [T_{\mu\nu}(x), T_{\lambda\rho}(0)] \rangle. \quad (3.2)$$

Thus finding the shear viscosity amounts to computing the zero-frequency limit of the imaginary part of the retarded correlator $G_{xyxy}(\omega, q)$. To compute the correlator, we follow the procedure outlined in [14] and used in [15] to determine the shear viscosity of the strongly coupled $\mathcal{N} = 4$ SYM at zero chemical potential. First, one has to determine the equation obeyed by the component $h_{xy}(u, t, z)$ of the gravitational perturbation of the background (2.5) - (2.7). By symmetry argument [15], [16] or by the direct analysis of perturbations of the equations of motion following from the Lagrangian (2.1) one can show that the Fourier component of the off-diagonal perturbation $\phi \equiv h_y^x$ decouples from all other perturbations and obeys the equation for a minimally coupled massless scalar in the background (2.5)

$$\phi_k'' + \frac{uf' - f}{uf} \phi_k' + \frac{\mathcal{H}\mathbf{w}^2 - f\mathbf{q}^2}{uf^2} \phi_k = 0, \quad (3.3)$$

where

$$\mathbf{w} = \frac{\omega}{2\pi T_0}, \quad \mathbf{q} = \frac{\omega}{2\pi T_0}. \quad (3.4)$$

The solution to eq. (3.3) in the hydrodynamic regime $\mathbf{w} \ll 1$, $\mathbf{q} \ll 1$ can be obtained along the lines of ref. [15]. We find

$$\phi_k = C_k f^{-i} \mathbf{w}^U \left\{ 1 - \frac{iU}{2} \left[\log \frac{c^3(u-1)^2}{a(u-1)^2 + b(u-1) + c} - \frac{b-2c}{\sqrt{b^2-4ac}} \log \frac{(2a(u-1) + b - \sqrt{b^2-4ac})(b + \sqrt{b^2-4ac})}{(2a(u-1) + b + \sqrt{b^2-4ac})(b - \sqrt{b^2-4ac})} \right] + \dots \right\}, \quad (3.5)$$

where ellipses denote higher order terms in the hydrodynamic expansion,

$$a = \prod_{i=1}^3 \kappa_i, \quad b = 2a - 1 - \sum_{i=1}^3 \kappa_i, \quad c = b - a - 1.$$

and

$$U(\kappa_1, \kappa_2, \kappa_3) = \frac{\sqrt{(1+\kappa_1)(1+\kappa_2)(1+\kappa_3)}}{2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1\kappa_2\kappa_3}.$$

In the limit $\kappa_i \rightarrow 0$ the solution (3.5) reduces to the one found in [15].

Another essential ingredient in computing the correlator is the boundary action. The total action is given by

$$S = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} d^5x \mathcal{L} + \frac{1}{8\pi G_5} \int_{\partial\mathcal{M}_5} d^4x \sqrt{-h} K + \frac{1}{8\pi G_5} \int_{\partial\mathcal{M}_5} d^4x \sqrt{-h} W, \quad (3.6)$$

where the second term is the Gibbons-Hawking boundary term, and the third term is required to make the action finite⁴ in the limit $u \rightarrow 0$. The explicit form of W for the background of interest was determined in [21]:

$$W = -\frac{\mathcal{H}^{1/3}}{L} \sum_{i=1}^3 H_i^{-1}. \quad (3.7)$$

Computing the action (3.6) on shell and expanding to quadratic order in h_x^y we find the following boundary action for ϕ_k

$$S_B = -\frac{\pi^2 N^2 T_0^4}{8} \frac{f(u)}{u} \phi'_k(u) \phi_{-k}(u) \Big|_0^1. \quad (3.8)$$

Following the prescription of [14] the retarded correlator is then computed as

$$G_{xyxy}(\omega, q) = -\frac{\pi^2 N^2 T_0^4}{4} \lim_{\epsilon \rightarrow 0} \frac{f(\epsilon) \phi'_k(\epsilon)}{\epsilon \phi_k(\epsilon)}. \quad (3.9)$$

(The solution $\phi_k(u)$ has been normalized to 1 at $u = \epsilon$.) We find

$$G_{xyxy}(\omega, q) = \frac{i\pi N^2 T_0^3 \omega c}{8} U(\kappa_1, \kappa_2, \kappa_3). \quad (3.10)$$

⁴For a discussion of relevant issues in the context of holographic renormalization, see e.g. [17–21].

The Kubo formula (3.1) gives the shear viscosity

$$\eta = \frac{\pi}{8} N^2 T_0^3 \sqrt{(1 + \kappa_1)(1 + \kappa_2)(1 + \kappa_3)} = \frac{\pi N^2 T_H^3 \prod_{i=1}^3 (1 + \kappa_i)^2}{(2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1 \kappa_2 \kappa_3)^3}. \quad (3.11)$$

Comparing this result with the expression (2.9) for the entropy density we immediately conclude that for any value of the chemical potential

$$\frac{\eta}{s} = \frac{1}{4\pi}. \quad (3.12)$$

For small κ_i we have

$$\eta = \frac{\pi N^2 T_H^3}{8} \left(1 + \frac{1}{2} \sum_{i=1}^3 \kappa_i + O(\kappa_i^2) \right). \quad (3.13)$$

4. Shear viscosity from the diffusion pole

In this Section we explicitly compute the retarded correlation functions of the stress-energy tensor and the R-currents and show that they exhibit a diffusion pole predicted by hydrodynamics. The value of the shear viscosity extracted from the pole is in agreement with eq. (3.11). For simplicity, in the rest of the paper we restrict ourselves to the case of a single charge black hole. We set $q_1 \neq 0$, $q_2 = q_3 = 0$ and omit the index “1” in all subsequent expressions.

4.1 The single charge black hole background

For a single charge black hole the effective Lagrangian (2.1) can be written as

$$\frac{\mathcal{L}}{\sqrt{-g}} = R + \frac{2}{L^2} \mathcal{V} - \frac{L^2}{8} H^{4/3} F^2 - \frac{1}{3} H^{-2} g^{\mu\nu} \partial_\mu H \partial_\nu H, \quad (4.1)$$

where $F_{\mu\nu}$ is the field-strength tensor of a $U(1)$ gauge field, and \mathcal{V} is the potential for the scalar field H ,

$$\mathcal{V} = 2H^{2/3} + 4H^{-1/3}. \quad (4.2)$$

The system of the gauged supergravity equations of motion for the fields $g_{\mu\nu}$, A_μ , H reads

$$\begin{aligned} \square H &= H^{-1} g^{\mu\nu} \partial_\mu H \partial_\nu H + \frac{L^2}{4} H^{7/3} F^2 - \frac{3}{L^2} H^2 \frac{\partial \mathcal{V}}{\partial H}, \\ \partial_\mu \left(\sqrt{-g} H^{4/3} F^{\mu\nu} \right) &= 0, \\ R_{\mu\nu} &= \frac{L^2}{4} H^{4/3} F_{\mu\gamma} F_\nu{}^\gamma + \frac{1}{3} H^{-2} \partial_\mu H \partial_\nu H - g_{\mu\nu} \left[\frac{2}{3L^2} \mathcal{V} + \frac{L^2}{24} H^{4/3} F^2 \right]. \end{aligned} \quad (4.3)$$

Consider small perturbations of the single charge background

$$g_{\mu\nu} = g_{\mu\nu}^0 + h_{\mu\nu}(u, t, z), \quad (4.4)$$

$$A_\mu = A_\mu^0 + A_\mu(u, t, z), \quad (4.5)$$

where $g_{\mu\nu}^0$, A_μ^0 are given by eqs. (2.5), (2.7) with $\kappa_1 \equiv \kappa$, $\kappa_2 = \kappa_3 = 0$. We assume that the perturbations depend on time, radial coordinate, and only one of the spatial world-volume coordinates, z . We are interested in gravitational fluctuations of the shear type, where the only nonzero components of $h_{\mu\nu}$ are h_{ta} , h_{za} , $a = x, y$. One can show that fluctuations of all other fields except $A_a(r, t, z)$, $a = x, y$, can be consistently set to zero. Introducing the new variables

$$T \equiv H_{ta} = g^{xx} h_{ta}, \quad Z \equiv H_{za} = g^{xx} h_{za}, \quad A = \frac{2A_a}{\mu} \quad (4.6)$$

the system of linearized equations derived from eqs. (4.3) can be written as

$$T' + \frac{\mathbf{q}f}{\mathbf{w}H} Z' + \frac{\kappa u}{2H} A = 0, \quad (4.7a)$$

$$T'' + \frac{uH' - H}{uH} T' - \frac{\mathbf{w}\mathbf{q}}{fu} Z - \frac{\mathbf{q}^2}{fu} T + \frac{\kappa u}{2H} A' = 0, \quad (4.7b)$$

$$Z'' + \frac{uf' - f}{uf} Z' + \frac{\mathbf{w}^2 H}{f^2 u} Z + \frac{\mathbf{q}\mathbf{w}H}{f^2 u} T = 0, \quad (4.7c)$$

$$(HfA' + 2(1 + \kappa)T)' - \frac{\mathbf{q}^2 H}{u} A + \frac{\mathbf{w}^2 H^2}{fu} A = 0. \quad (4.7d)$$

These equations are not independent: combining eq. (4.7a) with eq. (4.7b), one obtains eq. (4.7c). Thus it is sufficient to consider eqs. (4.7a), (4.7b), (4.7d). Expressing $Z(u)$ from eq. (4.7b), we differentiate it with respect to the radial coordinate u and substitute the resulting expression for $Z'(u)$ into eq. (4.7a). Thus we obtain a system of two coupled differential equations for $G(u) \equiv T'(u)$ and $A(u)$

$$G'' + \left(\frac{H'}{H} + \frac{f'}{f} \right) G' + \left(\frac{H'}{uH^2} - \frac{f'}{ufH} + \frac{\mathbf{w}^2 H}{uf^2} - \frac{\mathbf{q}^2}{uf} \right) G + \frac{\kappa u}{2H} A'' + \frac{\kappa u H f' + \kappa f(2H - uH')}{2fH^2} A' + \frac{\kappa \mathbf{w}^2}{2f^2} A = 0, \quad (4.8)$$

$$A'' + \left(\frac{f'}{f} + \frac{H'}{H} \right) A' + \frac{\mathbf{w}^2 H - \mathbf{q}^2 f}{uf^2} A + \frac{2(1 + \kappa)}{fH} G = 0. \quad (4.9)$$

(For $\kappa = 0$ eqs. (4.8), (4.9) decouple⁵ and reduce respectively to eqs. (6.15) and (5.5d) of ref. [15].) For the system (4.8), (4.9), the exponents at the singular point $u = 1$ corresponding to the horizon are

$$\alpha = \{ 0, 0, i\mathbf{w}U(\kappa), -i\mathbf{w}U(\kappa) \}, \quad (4.10)$$

where $U(\kappa) \equiv U(\kappa, 0, 0)$. Choosing the exponent $-i\mathbf{w}U(\kappa)$ corresponding to the incoming wave boundary condition at $u = 1$, we look for the solutions to eqs. (4.8), (4.9) in the hydrodynamic approximation in the form

$$G(u) = -\frac{u(2 + \kappa u)}{2H(u)^2} f^{-i\mathbf{w}U(\kappa)} (G_0(u) + \mathbf{w}G_1(u) + \mathbf{q}^2 G_2(u) + \dots), \quad (4.11)$$

$$A(u) = \frac{1}{\kappa H(u)} f^{-i\mathbf{w}U(\kappa)} (A_0(u) + \mathbf{w}A_1(u) + \mathbf{q}^2 A_2(u) + \dots), \quad (4.12)$$

⁵One should recall the rescaling (4.6).

where functions G_i , A_i are regular at $u = 1$. We obtain

$$G_0(u) = C_1 - \frac{C_2}{\kappa}, \quad (4.13)$$

$$A_0(u) = C_1 + uC_2, \quad (4.14)$$

where C_1, C_2 are the integration constants. Next,

$$G_1 = \frac{i}{\kappa \sqrt{\kappa+1} (\kappa+2) u (2+\kappa u)} \left\{ (u-1) \left(C_2 (\kappa+2) (2+\kappa u) - C_1 \kappa (2+\kappa+\kappa u+\kappa^2 u) \right) + 2(\kappa+1) (C_2 + uC_1) u (2+\kappa u) \log \frac{\kappa+2}{u+H(u)} \right\}, \quad (4.15)$$

$$A_1 = -\frac{i}{\sqrt{\kappa+1} (\kappa+2)} \left[(u-1) \left(C_1 \kappa - C_2 (\kappa+2) \right) + 2(\kappa+1) (C_1 + C_2 u) \log \frac{\kappa+2}{u+H(u)} \right]. \quad (4.16)$$

Functions $G_2(u)$, $A_2(u)$ are given by rather cumbersome expressions involving polylogarithms. We do not write them explicitly here.

The integration constants C_1, C_2 can be expressed in terms of the boundary values of the fields $T^{(0)}$, $Z^{(0)}$, $A^{(0)}$ by solving the equations

$$\lim_{u \rightarrow 0} A(u) = A^{(0)}, \quad (4.17)$$

$$\lim_{u \rightarrow 0} u f(u) \left(G' - \frac{1}{uH} G + \frac{\kappa u}{2H(u)} A' \right) = \mathbf{w} \mathbf{q} Z^{(0)} + \mathbf{q}^2 T^{(0)}, \quad (4.18)$$

where eq. (4.18) comes from eq. (4.7b). We find

$$C_1 = \kappa A^{(0)}, \quad (4.19)$$

$$C_2 = -\frac{2\kappa(1+\kappa)\mathbf{q}^2}{\mathbf{q}^2 - i2\sqrt{1+\kappa}\mathbf{w}} T^{(0)} + \frac{\kappa^2(\mathbf{q}^2 - i\sqrt{1+\kappa}\mathbf{w})}{\mathbf{q}^2 - i2\sqrt{1+\kappa}\mathbf{w}} A^{(0)}. \quad (4.20)$$

One can observe the appearance of the hydrodynamic pole in eq. (4.20).

4.2 The correlators

We are interested in two-point retarded correlation functions of stress-energy tensors and R -currents defined by

$$G_{\mu\nu\lambda\rho}(\omega, \mathbf{q}) = -i \int d^4x e^{-iq \cdot x} \theta(t) \langle [T_{\mu\nu}(x), T_{\lambda\rho}(0)] \rangle. \quad (4.21)$$

$$G_{\mu\nu\lambda}(\omega, \mathbf{q}) = -i \int d^4x e^{-iq \cdot x} \theta(t) \langle [T_{\mu\nu}(x), J_\lambda(0)] \rangle. \quad (4.22)$$

$$G_{\mu\nu}(\omega, \mathbf{q}) = -i \int d^4x e^{-iq \cdot x} \theta(t) \langle [J_\mu(x), J_\nu(0)] \rangle. \quad (4.23)$$

To compute the correlators we need to consider the boundary action. On shell, the action reduces to the surface terms, $S = S_{horizon} + S_\epsilon$, where

$$S_\epsilon = \lim_{u \rightarrow 0} \frac{\pi^2 N^2 T_0^4}{8} \int d^4x \left[-(1 + \kappa) - \frac{H}{u} H'_{ta} H_{ta} + \frac{f}{u} H'_{za} H_{za} + \frac{fH}{2\pi^2 T_0^2} A'_a A_a - \frac{3}{2}(1 + \kappa) H_{ta}^2 - \frac{1}{2}(1 + \kappa) H_{za}^2 + \frac{\sqrt{2\kappa(1 + \kappa)}}{2\pi T_0} H_{ta} A_a \right]. \quad (4.24)$$

The first term in eq. (4.24), $-\pi^2 N^2 T_0^4 (1 + \kappa) V_4 / 8$, is the density of the Gibbs potential Ω (i.e. the pressure with a minus sign), times the four-volume. The retarded two-point functions are obtained from S_ϵ following the recipe formulated in [14]. After substituting the solution (4.12) into eq. (4.24), the part of the boundary action quadratic in fluctuations assumes the form

$$S_\epsilon^{(2)} = \int \frac{d\omega dq}{(2\pi)^2} \phi_i^{(0)}(\omega, q) \mathcal{F}_{ik}(\omega, q) \phi_k^{(0)}(-\omega, -q), \quad (4.25)$$

where $\phi_i^{(0)}$ denote the boundary values of the fields $T^{(0)}$, $Z^{(0)}$, $A_a^{(0)}$. Then the retarded correlators are given by⁶

$$G^R = \begin{cases} -2\mathcal{F}_{ik}(\omega, q), & i = k, \\ -\mathcal{F}_{ik}(\omega, q), & i \neq k. \end{cases} \quad (4.26)$$

Computing the correlators, to leading order in the hydrodynamic approximation we obtain⁷

$$G_{tata}(\omega, q) = \frac{\sqrt{1 + \kappa} N^2 \pi T_0^3 q^2}{8(i\omega - \mathcal{D}q^2)}, \quad (4.27)$$

$$G_{taza}(\omega, q) = -\frac{\sqrt{1 + \kappa} N^2 \pi T_0^3 \omega q}{8(i\omega - \mathcal{D}q^2)}, \quad (4.28)$$

$$G_{zaza}(\omega, q) = \frac{\sqrt{1 + \kappa} N^2 \pi T_0^3 \omega^2}{8(i\omega - \mathcal{D}q^2)}, \quad (4.29)$$

where the diffusion constant is given by

$$\mathcal{D} = \frac{1}{4\pi T_0 \sqrt{1 + \kappa}} = \frac{1}{4\pi T_H} \frac{1 + \kappa/2}{1 + \kappa}. \quad (4.30)$$

In the limit $\kappa \rightarrow 0$ the results (4.27) - (4.30) reduce to those obtained in [15] for the case of a zero chemical potential. For the correlators of the components of a stress-energy tensor and an R -current we have

$$G_{taa}(\omega, q) = \frac{i\sqrt{2\kappa(1 + \kappa)} N^2 \pi T_0^3 \omega}{8(i\omega - \mathcal{D}q^2)}. \quad (4.31)$$

$$G_{zaa}(\omega, q) = -\frac{\sqrt{2\kappa} N^2 T_0^2 \omega q}{32(i\omega - \mathcal{D}q^2)}. \quad (4.32)$$

⁶Note that we obtain the correlators with upper indices. Indices of the boundary theory correlators are raised or lowered with the flat Minkowski metric, so that e.g. $G^{taza} = -G_{taza}$.

⁷Contact terms are ignored.

These correlators vanish in the limit $\kappa \rightarrow 0$, in agreement with [15]. Finally, the retarded correlator of the x (or y) component of the R-currents is given by

$$G_{xx}(\omega, q) = G_{yy}(\omega, q) = G_{aa}(\omega, q) = \frac{i\kappa N^2 T_0^2 \omega}{16(i\omega - \mathcal{D}q^2)} + O(\omega, q^2). \quad (4.33)$$

In the limit $\kappa \rightarrow 0$ the leading contribution in eq. (4.33) vanishes. The subleading term gives $G_{aa} = -iN^2 T_0 \omega / 16\pi$ which coincides with the result obtained in [15].

In the limit of vanishing spatial momentum the nontrivial contribution to $G_{aa}(\omega, q)$ again comes from the subleading term in eq. (4.33). It is given by

$$G_{aa}(\omega, 0) = -\frac{i(\kappa + 2)^2 N^2 T_0 \omega}{64\pi\sqrt{\kappa + 1}}. \quad (4.34)$$

We also find that in the limit $q \rightarrow 0$ the correlators G_{taa} and G_{tata} vanish (modulo contact terms).

4.3 The diffusion pole

All the retarded correlators in the shear channel exhibit a diffusion pole with the dispersion relation

$$\omega = -i\mathcal{D}q^2, \quad (4.35)$$

where the diffusion constant \mathcal{D} is given by eq. (4.30). To find viscosity, recall that in hydrodynamics

$$\mathcal{D} = \frac{\eta}{\varepsilon + P}.$$

From thermodynamics it follows that

$$\varepsilon + P = T_{Hs} + \mu\rho = \frac{2(1 + \kappa)}{2 + \kappa} T_{Hs}.$$

Thus for the ratio of shear viscosity to entropy density we find $\eta/s = 1/4\pi$ which coincides with the result (3.12) obtained from the Kubo formula.

5. Thermal conductivity

Thermal conductivity κ_T can be computed using the appropriate Kubo formula (see Appendix B)

$$\kappa_T = -\frac{(\varepsilon + P)^2}{\rho^2 T} \lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im} G(\omega, \mathbf{0}). \quad (5.1)$$

Here G is the retarded Green's function of the R-current components J^x given by eq. (4.34). Thus we find

$$\kappa_T = \frac{N^2(\kappa + 2)}{32\pi} \frac{(\varepsilon + P)^2}{\rho^2} = \pi N^2 T_H^2 \frac{(1 + \kappa)^2}{\kappa(\kappa + 2)}. \quad (5.2)$$

In terms of the chemical potential \mathfrak{m} the thermal conductivity can be written as

$$\kappa_T = \pi N^2 T_H^2 \frac{1 + \sqrt{1 - 4\mathfrak{m}^2} - \mathfrak{m}^2(\sqrt{1 - 4\mathfrak{m}^2} - 5)}{8\mathfrak{m}^2}. \quad (5.3)$$

Comparing this result to the one for the shear viscosity (3.11), we observe that for a single-charge black hole, the shear viscosity and thermal conductivity can be expressed in terms of the chemical potential $\mu_1 \equiv \mu$ as

$$\eta = \frac{\pi N^2 T_H^3}{8} F_\eta(\mu, T_H), \quad \kappa_T = \frac{\pi N^2 T_H^2}{8} F_\kappa(\mu, T_H), \quad (5.4)$$

where the functions F_η, F_κ depend only on the ratio $\mathbf{m} = \mu/2\pi T_H$,

$$F_\eta(\mu, T_H) = \frac{8\mathbf{m}^2 \left(1 - \sqrt{1 - 4\mathbf{m}^2} - \mathbf{m}^2\right)^2}{\left(1 - \sqrt{1 - 4\mathbf{m}^2}\right)^3}, \quad F_\kappa(\mu, T_H) = \frac{2}{\mathbf{m}^2} F_\eta(\mu, T_H). \quad (5.5)$$

Thus for all values of T_H and μ one finds an analogue of the Wiedemann-Franz law [24]

$$\frac{\kappa_T \mu^2}{\eta T_H} = 8\pi^2. \quad (5.6)$$

For small \mathbf{m} we get

$$F_\eta = 1 + \mathbf{m}^2 - \mathbf{m}^6 + O(\mathbf{m}^8).$$

The function $F_\eta(\mathbf{m})$ is shown in Fig. 1.

6. Critical behavior of transport coefficients

The boundary of thermodynamic stability is $\mathbf{m}_c = 1/2$ or $\mu_c = \pi T$. Expanding the shear viscosity and the thermal conductivity near $\mathbf{m} = \mathbf{m}_c$ we obtain

$$\eta = \eta_* \left[1 + \frac{2}{3} \sqrt{\mathbf{m}_c - \mathbf{m}} - \frac{20}{9} (\mathbf{m}_c - \mathbf{m}) + O\left((\mathbf{m}_c - \mathbf{m})^{3/2}\right) \right], \quad (6.1)$$

$$\kappa_T = \kappa_{T*} \left[1 + \frac{2}{3} \sqrt{\mathbf{m}_c - \mathbf{m}} + \frac{16}{9} (\mathbf{m}_c - \mathbf{m}) + O\left((\mathbf{m}_c - \mathbf{m})^{3/2}\right) \right], \quad (6.2)$$

where

$$\eta_* = \frac{9\pi N^2 T_H^3}{64}, \quad \kappa_{T*} = \frac{9\pi N^2 T_H^2}{8}.$$

Both the viscosity and the thermal conductivity are finite⁸ at the critical point. Their derivatives diverge with the critical index equal to 1/2.

7. Conclusion

We have considered the hydrodynamic regime of the $\mathcal{N} = 4$ supersymmetric theory at finite temperature and finite chemical potential. We have computed the shear viscosity and the thermal conductivity. The shear viscosity is computed using two different methods which give the same answer. We find that the ratio of shear viscosity and the entropy density

⁸Curiously, the shear viscosity of He^4 near the λ -point is also finite and its first derivative is divergent, as first shown theoretically by A. M. Polyakov [25] and later confirmed experimentally [26].

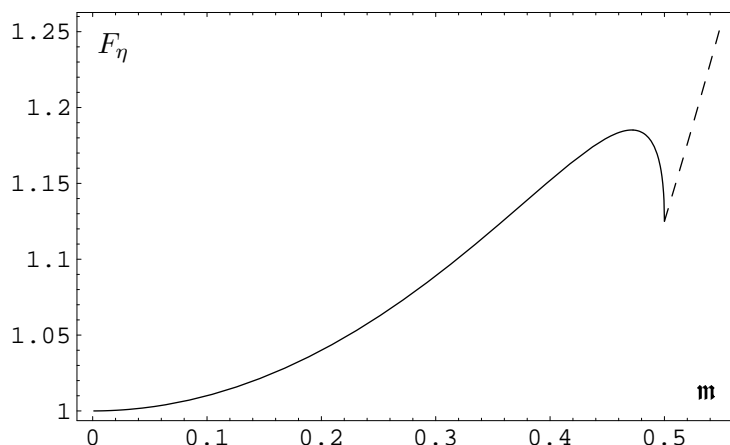


Figure 1: Normalized shear viscosity F_η as a function of the chemical potential $\mathfrak{m} = \mu/2\pi T_H$. There is a cusp singularity at $\mu_c = \pi T_H$. The part of the curve to the right of the singularity (shown in dashed line) is unphysical.

is always equal to $1/4\pi$, which is the same value found so far in all theories with gravity duals. Our result demonstrates that the universality of this ratio extends to theories with gravity duals at finite chemical potentials. We found a curious relationship between the shear viscosity and the thermal conductivity similar to the Wiedemann-Franz law, and we have also determined the critical behavior of the kinetic coefficients near the boundary of thermodynamic stability.

One possible extension of this work is to compute the thermal conductivity and diffusion coefficients in the case when all three conserved charges are nonzero. This would complete our knowledge of the kinetic coefficients of $\mathcal{N} = 4$ super-Yang-Mills theory in the whole phase diagram, as the conformal invariance of the theory guarantees that the speed of sound is equal to $1/\sqrt{3}$ and the bulk viscosity is zero for any temperature and chemical potential.

Note added: while this paper was being completed, we became aware of the work on the same subject by J. Mas [27] whose results are in agreement with our analysis. Another recent paper on the subject is by K. Maeda, M. Natsuume, and T. Okamura [28]. A closely related work on the hydrodynamics of M2-branes by O. Saremi appeared in [29].

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A. Rescaling the black hole solution

The metric (2.5) of the gravity background dual to $\mathcal{N} = 4$ SYM with nonzero chemical potentials in Minkowski space can be obtained by rescaling and taking an infinite volume limit of the black hole solution with a spherical horizon. (At zero chemical potential, this procedure leads from the metric corresponding to an AdS-Schwarzschild black hole to the metric describing the near-horizon region of non-extremal three-branes.) Following refs. [22, 9], here we explain how the rescaling works for the solution itself as well as for various thermodynamic quantities associated with it.

Focusing for simplicity on a single charge black hole, the metric (2.2) with $k = 1$ reads

$$ds_5^2 = -H^{-2/3} f_1 dt^2 + H^{1/3} (f_1^{-1} dr^2 + r^2 d\Omega_3^2), \quad (\text{A.1})$$

where

$$f_1 = 1 - \frac{m_1}{r^2} + \frac{r^2}{L^2} H, \quad H = 1 + \frac{q}{r^2}.$$

Rescaling

$$r \rightarrow \lambda^{1/4} r, \quad t \rightarrow \lambda^{-1/4} t, \quad m_1 \rightarrow \lambda m_1, \quad q \rightarrow \lambda^{1/2} q, \quad (\text{A.2})$$

and taking $\lambda \rightarrow \infty$ while simultaneously blowing up the sphere

$$L^2 d\Omega_3^2 \rightarrow \lambda^{-1/2} (dx^2 + dy^2 + dz^2) \quad (\text{A.3})$$

in the limit we obtain the metric with a translationally invariant horizon

$$ds_5^2 = -H^{-2/3} \frac{r^2}{L^2} f dt^2 + H^{1/3} \frac{L^2}{r^2 f} dr^2 + H^{1/3} \frac{r^2}{L^2} (dx^2 + dy^2 + dz^2), \quad (\text{A.4})$$

where $f = 1 + q/r^2 - r_0^4/r^4$, with $r_0^4 \equiv m_1 L^2$.

Similar reasoning applies to thermodynamic quantities and their densities. Temperature, entropy, energy and the Gibbs potential scale as

$$T_H \rightarrow \lambda^{1/4} T_H, \quad S \rightarrow S, \quad E \rightarrow \lambda^{1/4} E, \quad \Omega \rightarrow \lambda^{1/4} \Omega. \quad (\text{A.5})$$

For the background (A.1), the inverse Hawking temperature, entropy, energy, and the thermodynamic potential Ω are given correspondingly by (see e.g. [21])

$$\begin{aligned} \beta &= 2\pi L^2 \frac{(r_+^2 + q)^{1/2}}{1 + q + 2r_+^2}, & S_{BH} &= \frac{\pi^2}{2G_5} r_+^2 (r_+^2 + q)^{1/2}, \\ E &= \frac{\pi}{G_5} \left(\frac{3r_0^4}{8L^2} + \frac{q}{4} + \frac{3L^2}{32} \right), & \Omega &= \frac{\pi}{G_5} \left(-\frac{r_0^4}{8L^2} + \frac{r_+^2}{4} + \frac{3L^2}{32} \right). \end{aligned}$$

In the limit $\lambda \rightarrow \infty$ for the rescaled quantities we find

$$\begin{aligned} \beta &= 2\pi L^2 \frac{(r_+^2 + q)^{1/2}}{q + 2r_+^2}, & S_{BH} &= \lambda^{3/4} \frac{\pi^2}{2G_5} r_+^2 (r_+^2 + q)^{1/2}, \\ E &= \lambda^{3/4} \frac{\pi}{G_5} \frac{3r_0^4}{8L^2}, & \Omega &= -\lambda^{3/4} \frac{\pi}{G_5} \frac{r_0^4}{8L^2}. \end{aligned} \quad (\text{A.6})$$

From (A.3) one can see that $2\pi^2 L^3 \rightarrow \lambda^{-3/4} V_3$ and thus the pressure and the densities of entropy and energy are finite in the $\lambda \rightarrow \infty$ limit and are given by

$$s = S_{BH}/V_3 = \frac{r_+^2 (r_+^2 + q)^{1/2}}{4G_5 L^3}, \quad (\text{A.7})$$

$$\varepsilon = E/V_3 = \frac{3r_0^4}{16\pi G_5 L^5}, \quad (\text{A.8})$$

$$P = -\Omega/V_3 = \frac{r_0^4}{16\pi G_5 L^5}. \quad (\text{A.9})$$

With identifications⁹ (2.6) and $G_5 = \pi L^3/2N^2$ from eqs. (A.7)–(A.9) we obtain the expressions (2.9)–(2.11) used in the main text.

B. Relativistic hydrodynamics at finite chemical potential

For completeness here we review the hydrodynamics of a relativistic fluid with one conserved charge. The hydrodynamic equations include the continuity equations

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu J^\mu = 0 \quad (\text{B.1})$$

and the constitutive equations, which formally have the form

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}, \quad J^\mu = \rho u^\mu + \nu^\mu \quad (\text{B.2})$$

Here ε and P are the local energy density and pressure, u^μ is the local velocity, $u_\mu u^\mu = -1$. The parts $\tau^{\mu\nu}$ and ν^μ are the dissipative parts of the stress-energy tensor and the current. To complete the system of equations we need expressions relating $\tau^{\mu\nu}$ and ν^μ with derivatives of u^μ and of the thermodynamic potentials.

Following Landau and Lifshitz [23], we can choose u^μ and ρ so that $\tau^{\mu\nu}$ and ν^μ are orthogonal to u^μ

$$u_\mu \tau^{\mu\nu} = u_\mu \nu^\mu = 0. \quad (\text{B.3})$$

The most general form of the constitutive equation follows from the second law of thermodynamics. First we notice that

$$u_\nu \partial_\mu T^{\mu\nu} = -(\varepsilon + P)\partial_\mu u^\mu - u^\mu \partial_\mu \varepsilon + u_\nu \partial_\mu \tau^{\mu\nu} = 0 \quad (\text{B.4})$$

Using the thermodynamic relations

$$\varepsilon + P = Ts + \mu\rho, \quad d\varepsilon = Tds + \mu d\rho, \quad (\text{B.5})$$

current conservation, and eq. (B.3), eq. (B.4) can be transformed into

$$\partial_\mu (s u^\mu) = \frac{\mu}{T} \partial_\mu \nu^\mu - \frac{\tau^{\mu\nu}}{T} \partial_\mu u_\nu \quad (\text{B.6})$$

⁹Note that $r_0^4 = r_+^2(r_+^2 + q)$, where r_+ is the largest root of the equation $f(r) = 0$.

or

$$\partial_\mu \left(s u^\mu - \frac{\mu}{T} \nu^\mu \right) = -\nu^\mu \partial_\mu \frac{\mu}{T} - \frac{\tau^{\mu\nu}}{T} \partial_\mu \nu^\mu. \quad (\text{B.7})$$

We now interpret the left hand side as the divergence of the entropy current. The right hand side thus must be positive. This implies

$$\nu^\mu = -\varkappa \left(\partial^\mu \frac{\mu}{T} + u^\mu u^\lambda \partial_\lambda \frac{\mu}{T} \right), \quad (\text{B.8})$$

$$\tau^{\mu\nu} = -\eta (\partial^\mu u^\nu + \partial^\nu u^\mu + u^\mu u^\lambda \partial_\lambda u^\nu + u^\nu u^\lambda \partial_\lambda u^\mu) - \left(\zeta - \frac{2}{3} \eta \right) (g^{\mu\nu} + u^\mu u^\nu) \partial_\lambda u^\lambda. \quad (\text{B.9})$$

Here η and ζ are the shear and bulk viscosities, respectively. To have an interpretation of \varkappa as the coefficient of thermal conductivity, let us consider the case when there is no charge transport, $J^i = 0$, but there is an energy flow, $T^{ti} \neq 0$, which is the heat flow. The local velocity u^i is necessarily small and is equal to

$$u^i = \frac{\varkappa}{\rho} \partial^i \frac{\mu}{T}. \quad (\text{B.10})$$

Therefore

$$T^{ti} = (\epsilon + P) u^i = \frac{\varkappa}{\rho} \partial^i \frac{\mu}{T} (\epsilon + P). \quad (\text{B.11})$$

Using $dP = s dT + \rho d\mu$ one can write this equation as

$$T^{ti} = -\varkappa \left(\frac{\epsilon + P}{\rho T} \right)^2 \left(\partial_i T - \frac{T}{\epsilon + P} \partial_i P \right). \quad (\text{B.12})$$

In the nonrelativistic theory the heat flow is proportional to the gradient of temperature; in the relativistic limit there is an extra contribution proportional to the gradient of pressure. The proportionality coefficient is the thermal conductivity,

$$\kappa_T = \left(\frac{\epsilon + P}{\rho T} \right)^2 \varkappa. \quad (\text{B.13})$$

The Kubo's formula for κ_T can be written down as follows. Suppose one puts the thermal system in a slowly-varying external background gauge field A^μ coupled to the conserved charge. This field will induce a current, proportional to the electric field $E_i = \partial_t A_i - \partial_i A_t$. But since A_t plays the same role as the chemical potential, by comparing with eq. (B.8) we can write

$$J^i = \frac{\varkappa}{T} (\partial^t A^i - \partial^i A^t) \quad (\text{B.14})$$

In the case when the external fields are homogeneous in space, this relation becomes very simple,

$$J^i = i \frac{\varkappa}{T} \omega A^i. \quad (\text{B.15})$$

We can compare this relation with the one that follows from the linear response theory, $J^i = -G_R(\omega, \mathbf{0}) A^i$. Thus we obtain

$$G_R(\omega, \mathbf{0}) = -i \omega \frac{\varkappa}{T}. \quad (\text{B.16})$$

From eq. (B.13) we find the Kubo formula (5.1).

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